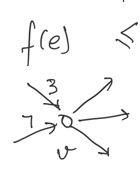
Max Flows 1: Min Cuts

Friday, 29 September 2023 12:19 AM

MAX - FLOWS :

Given a directed graph $G = (V, E)$, spl. vertices s, t
s: source, t: sink. Additionally, $c_e \in \mathbb{Z}_+$ for each edge $e \in E$

An s-t $f : E \to \mathbb{R}_+$ satisfies the following:
(i) $\forall e \in E \quad f(e) \leq c_e$
(ii) 

$\forall v \neq s,t \quad \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

Value of a flow f, also denote $|f| = \sum_{e \text{ out of } s} f(e)$

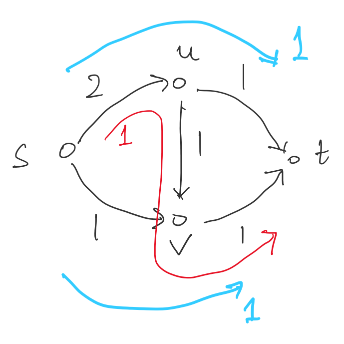
(assume that no flow going into s, i.e., $\forall e = (v,s), f(e) = 0$)

Note that since $\sum_v \underbrace{\left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ into } v} f(e) \right)}_{= 0 \ \forall v \neq s,t} = 0$

$$\Rightarrow \left[\sum_{e \text{ out of } s} f(e) = \sum_{e \text{ into } t} f(e) \right] = |f|$$

(assume that no flow going out of t)

Example 1:



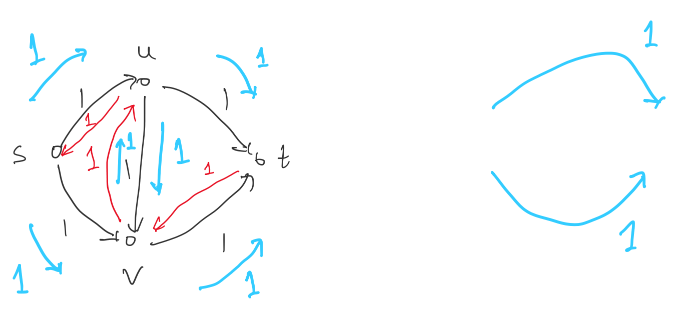
f: $f(s,u) = 1$
$f(u,v) = 1$
$f(v,t) = 1$
$|f| = 1$

f':
$f'(s,u) = 2$
$f'(u,t) = 2$
is not a flow

g: $g(s,u) = g(u,t) = 1$ $\quad |g| = 2$
$g(s,v) = g(v,t) = 1$

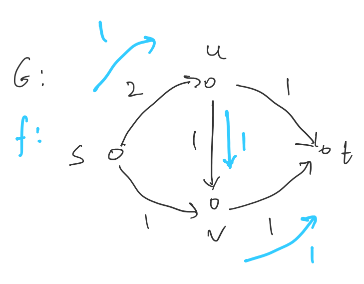
h: $h(s,u) = 2 \quad h(u,t) = 1 \quad |h| = 2$
$h(u,v) = h(v,t) = 1$

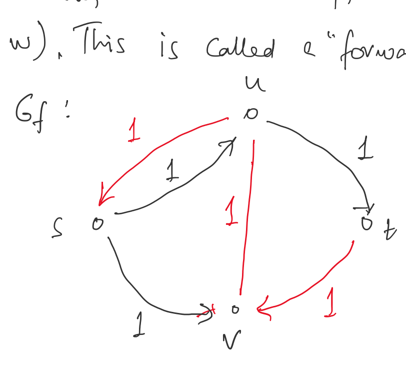
Given a graph $G = (V, E)$, source s, sink t, edge capacities c_e, the MAX s-t FLOW problem is to find an s-t flow of maximum value.



RESIDUAL GRAPH :

Given a graph $G = (V, E)$, capacities c_e, and an s-t flow f. The residual graph $G_f = (V, E_f)$ has as edges
(i) If $(v,w) \in E$, $f(v,w) > 0$, then $(w,v) \in E_f$, $c^f_{wv} = f_{vw}$
Such an edge is called a "backward" edge
(ii) If $(v,w) \in E$, $f(v,w) < c_{vw}$, then $(v,w) \in E_f$,
$c^f_{vw} = c_{vw} - f(v,w)$. This is called a "forward" edge.



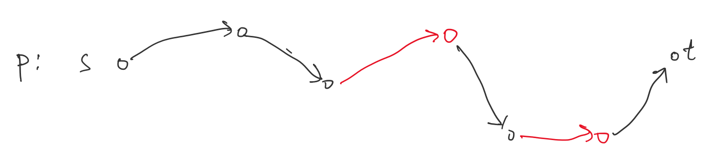


Black: fwd edges
Red: Reverse or backward edges

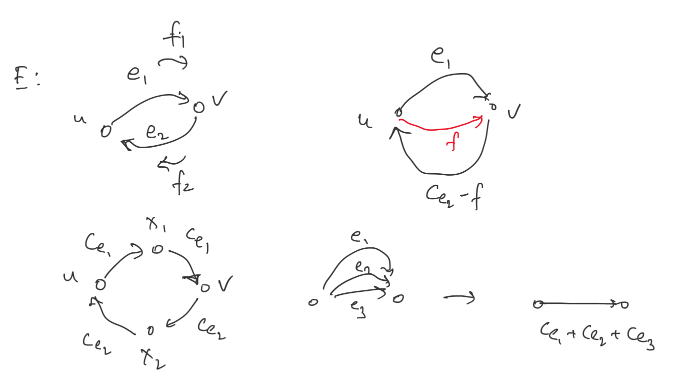
Claim: If f is a maximum flow, then G_f does not have an s-t path.

Proof: Will show that if there is an s-t path in G_f then f is not a max flow.
Let $p = (s = v_0, v_1, \ldots, v_{k-1}, v_k = t)$ be an s-t path in G_f.
Let $\delta = \min_{e \in p} c_e > 0$

Consider the following flow h:
$\forall e = (u,w) \in p$: (i) if (u,w) is a fwd edge,
$h(u,w) = f(u,w) + \delta$
(ii) if (u,w) is a backward edge,
$h(w,u) = f(w,u) - \delta$
(since in original graph, $f(w,u) \geq \delta$)
& $h(e) = f(e) \quad \forall e \notin p$
Check yourself: h is a flow
$|h| = |f| + \delta > |f|$ ∎



Parallel edges:



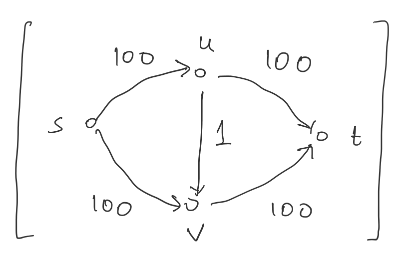
Ford-Fulkerson $(G = (V,E), c_e, s, t)$
Initially $f_e = 0 \quad \forall e \in E \quad (G_f = G)$
While $\exists$ s-t path p in G_f
Let $\delta = \min_{e \in p} \{c^f_e\}$
Push flow of value δ along path p, i.e.,

$$f(u,v) = \begin{cases} f(u,v) + \delta & \text{if } (u,v) \in p \text{ is a fwd edge} \\ f(v,u) - \delta & \text{if } (u,v) \in p \text{ is a bwd edge} \\ f(u,v) & \text{if } (u,v) \notin p \end{cases}$$

Note: (i) in every iteration $|f| = \sum_{e \text{ out of } s} f_e$ increases by δ. (prove formally)

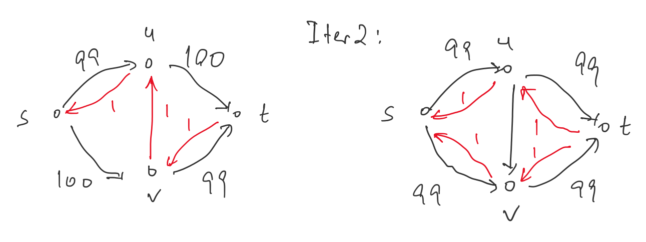
(ii) assuming all capacities are positive integers, in any iteration, f_e is integral $\forall e \in E$, and hence $\delta \geq 1$

From (ii) # of iterations is $O(|f_{max}|)$
Consider:



Clearly $|f_{max}| = 200$

After iteration 1: Iter 2:

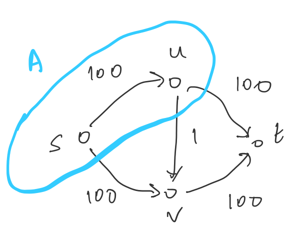


... if we always choose path that includes (u,v) or (v,u) edge, then F-F takes 200 iterations to find max-flow.

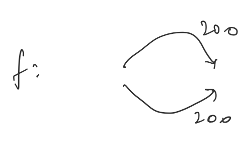
Thus, F-F algo is NOT a poly-time algo. It is however a pseudo polynomial-time algo.

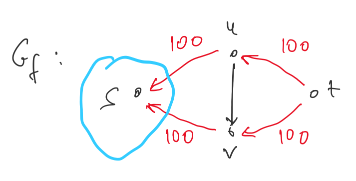
CUTS: A cut in a graph is a bipartition of the vertices $(A, V \setminus A)$ where $A \subseteq V$.
An s-t cut in a graph is a bipartition $(A, V \setminus A)$ where $s \in A$, $t \in V \setminus A$
Capacity of a cut $(A, V \setminus A) = \sum_{\substack{e = (u,v): \\ u \in A \\ v \notin A}} c_e$

A min s-t cut is an s-t cut of minimum capacity



min s-t cut for this graph has capacity 200
Will use $C_G(A)$ to denote capacity of cut $(A, V/A)$ in graph G





Theorem: Max-Flow Min-Cut: Given a graph $G = (V, E)$, $c_e \in \mathbb{Z}_{++}$, $s, t \in V$, the following statements are equivalent:
(i) f is a maximum flow
(ii) there is no s-t path in G_f
(iii) $\exists$ an s-t cut of capacity $|f|$

Proof: (i) ⇒ (ii): already shown in previous claim
(iii) ⇒ (i): Let f be s-t flow, & $(A, V/A)$ be any s-t cut.

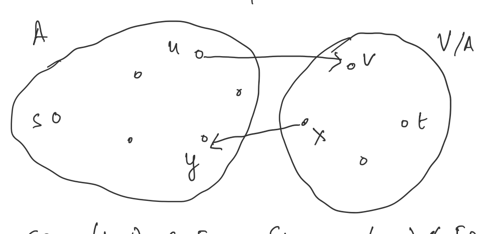
$$|f| = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \leq \sum_{e \text{ out of } A} f(e) \leq \sum_{e \text{ out of } A} c_e$$

uses flow conservation — uses capacity constraints

$= \text{capacity of cut } A$
$= C_G(A)$

Hence, if $\exists$ flow f & s-t cut $(A, V \setminus A)$ s.t. $|f| = C_G(A)$, then f is a max-flow (& A is a min s-t cut).

(ii) ⇒ (iii) Let $A \subseteq V$ be the set of vertices reachable from s in the residual graph G_f. Note that $t \notin A$.
Claim: $|f| = \text{Cut}_G(A)$ (note: $\text{Cut}_{G_f}(A) = 0$)
Proof: $|f| = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$



say $(u,v) \in E$. Since $(u,v) \notin E_f$, this edge must be saturated, hence $f(u,v) = c_{uv}$
Thus, $|f| = \sum_{e \text{ out of } A} c_e - \sum_{e \text{ into } A} f(e)$

Say $(x,y) \in E$. Then $f(x,y) = 0$, o.w. there would be a backward (y,x) edge in E_f
& hence $|f| = \sum_{e \text{ out of } A} c_e = C_G(A)$ ∎

This completes proof of the max-flow min-cut theorem.